Math 217 Fall 2025
Quiz 14 – Solutions

Dr. Samir Donmazov

1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Suppose U is a vector space and $u_1, u_2, \ldots, u_n \in U$. The *span* of $(u_1, u_2, \ldots, u_n)$ is ...

   > **Solution:** The set of all linear combinations of the vectors:
   >
   > $$\operatorname{Span}(u_1, \ldots, u_n) \;=\; \{\, a_1 u_1 + \cdots + a_n u_n \mid a_1, \ldots, a_n \in \mathbb{F} \,\},$$
   >
   > where $\mathbb{F}$ is the underlying field (e.g. $\mathbb{R}$ or $\mathbb{C}$).

   (b) A *subspace* of a vector space V is ...

   > **Solution:** A nonempty subset $W \subseteq V$ such that:
   >
   > - $0_V \in W$;
   > - if $w_1, w_2 \in W$, then $w_1 + w_2 \in W$ (closed under addition);
   > - if $\alpha \in \mathbb{F}$ and $w \in W$, then $\alpha w \in W$ (closed under scalar multiplication).
   >
   > Furthermore, W is a vector space under the operations inherited from V.

   (c) To say that a list of vectors $(x_1, x_2, \ldots, x_d)$ in a vector space X is *linearly independent* means ...

   > **Solution:** The only scalars $a_1, \ldots, a_d \in \mathbb{F}$ satisfying
   >
   > $$a_1 x_1 + \cdots + a_d x_d = 0_X$$
   >
   > are $a_1 = \cdots = a_d = 0$. Equivalently, no x_j can be written as a linear combination of the others.

2. Let V be a vector space, and let $\mathcal{S} = (w_1, w_2, \ldots, w_m)$ be a list of vectors in V that spans V. Suppose that some $w_j \in \mathcal{S}$ is in the span of the list $\mathcal{S} \setminus \{w_j\} := (w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_m)$. Prove that $\mathcal{S} \setminus \{w_j\}$ also spans V.

*For full credit, please write out fully what you mean instead of using shorthand phrases.

Solution: Because $\mathcal{S}$ spans V, for any $v \in V$ there exist scalars $a_1, \dots, a_m \in \mathbb{F}$ such that

$$v = a_1 w_1 + \cdots + a_{j-1} w_{j-1} + a_j w_j + a_{j+1} w_{j+1} + \cdots + a_m w_m.$$

By hypothesis w_j is a linear combination of the other vectors, i.e., there exist scalars c_i ($i \neq j$) with

$$w_j = \sum_{i \neq j} c_i \, w_i.$$

Substitute this into the expression for v:

$$v = \sum_{i \neq j} a_i w_i + a_j \Big(\sum_{i \neq j} c_i \, w_i \Big) = \sum_{i \neq j} \big(a_i + a_j c_i \big) \, w_i.$$

Thus v is a linear combination of the vectors in $\mathcal{S} \setminus \{w_j\}$. Since $v \in V$ was arbitrary, $\mathcal{S} \setminus \{w_j\}$ spans V.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) Let V be a vector space, and let $\mathcal{S}$ be a linearly independent list of vectors in V. For any $v \in V \setminus \mathcal{S}$, we have that $\mathcal{S} \cup \{v\}$ is linearly independent *and* $v \in \operatorname{Span}(\mathcal{S})$.

Solution: FALSE. Both assertions cannot hold simultaneously for an arbitrary v. Counterexamples:

- Take $V = \mathbb{R}^2$, $\mathcal{S} = \{[1, 0]^T\}$, and $v = [0, 1]^T$. Then $\mathcal{S} \cup \{v\}$ is linearly independent, but $v \notin \operatorname{Span}(\mathcal{S})$.
- Take $V = \mathbb{R}^2$, $\mathcal{S} = \{[1, 0]^T\}$, and $v = [2, 0]^T$. Then $v \in \operatorname{Span}(\mathcal{S})$, but $\mathcal{S} \cup \{v\}$ is linearly *dependent*.

In general, $\mathcal{S} \cup \{v\}$ is linearly independent iff $v \notin \operatorname{Span}(\mathcal{S})$.

(b) The kernel of the matrix

$$A = \begin{bmatrix} 1 & -21 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is two dimensional.

Solution: FALSE. Solve $A\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = (x_1, x_2, x_3)^T$:

$$\begin{cases} x_1 - 21 x_2 = 0 \;\Rightarrow\; x_1 = 21 x_2, \\ x_3 = 0. \end{cases}$$

Let $t := x_2$ be free. Then $\mathbf{x} = t\,(21, 1, 0)^T$. The solution set is a one-dimensional subspace of $\mathbb{R}^3$, so $\ker A$ is 1-dimensional, not 2-dimensional.